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The Representation of Numbers

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ABSTRACT

This article explores the representational structures of numeration systems and the cognitive factors of the representational effect in numerical tasks, focusing on external representations and their interactions with internal representations. Numeration systems are analyzed at four levels: dimensionally, dimensional representations, bases, and symbol representations. The representational properties at these levels affect the processes of numerical tasks in different ways and are responsible for different aspects of the representational effect. This hierarchical structure is also a cognitive taxonomy that can classify nearly all numeration systems that have been invented across the world. Multiplication is selected as an example to demonstrate that complex numerical tasks require the interwoven processing of information distributed across internal and external representations. Finally, a model of distributed numerical cognition is proposed and an answer to the question of why Arabic numerals are so special is provided.

We all know that Arabic[1] numerals are more efficient than Roman and many other types of numerals for calculation (e.g., 73 × 27 is easier than LXXIII × XXVII), even though they all represent the same entities—numbers (see Figure 1 for several examples of numeration systems). This *representational effect*, that different representations of a common abstract structure can cause dramatically different cognitive behaviors, has had profound significance in the development of arithmetic and algebra in particular and mathematics in general. The Arabic numeration system, remarkable as is its simplicity, has been regarded as one of the greatest inventions of the human mind.

The representational effect of numeration systems is a cognitive phenomenon. However, early studies of numeration systems focused on their historical, cultural, mathematical, and philosophical aspects (e.g., Brooks, 1876; Cajori, 1928; Dantzig, 1939; Flegg, 1983; Ifrah, 1987; Menninger, 1969). Recently, the representational efficiencies of different numeration systems have been compared and analyzed in terms of their cognitive properties (e.g., Nickerson, 1988; Norman, 1993), and the nature of number representations has also been examined from a semiotic perspective (Becker & Varelas, 1993). In this paper, we attempt a systematic analysis of the representational structures of numeration systems and the cognitive factors responsible for the representational effect in numerical tasks.

Unlike the representational effect of numeration systems, there have been a large number of psychological studies of numerical cognition over the last two decades (for a recent collection of re-

[1] Although the Arabic numerals were originally invented in India and are called Hindu-Arabic numerals by some historians, we adopt the conventional name for simplicity.

views, see Dehaene, 1993). However, most of these studies have entirely focused on internal representations: how people perform numerical tasks in their heads, how numbers and arithmetic facts are represented in memory, and what mental processes and procedures are involved in the comprehension, calculation, and production of numbers. In this present study, we focus on external representations and their interactions with internal representations: how numbers are represented in the external environment, what structures in external representations are perceived and processed, and how internal and external representations are integrated in numerical tasks. Although our focus is on external representations, we are not denying the important roles of internal representations. What we attempt to show is that external representations have much more important roles than previously acknowledged.

We start with a representational analysis of the hierarchical structures of numeration systems. From this analysis, we develop a cognitive taxonomy of numeration systems. This taxonomy can not only classify most numeration systems but also can serve as a theoretical framework for systematic studies of the representations of numbers and the processes in numerical tasks. In the second part, we analyze how the dimensional representations of numeration systems are distributed across internal and external representations. In the third part, we use multiplication as an example to examine the interwoven processing of internal and external information in complex numerical tasks. In the last part, we outline a model of distributed numerical cognition and provide an answer to the question of why Arabic numerals are so special.

THE REPRESENTATIONAL STRUCTURES OF NUMERATION SYSTEMS

In this part, we first analyze the dimensionality of numeration systems. Next, we analyze the representational properties of numeration systems at four different levels in terms of a hierarchical structure. Then, based on the hierarchical structure, we propose . a cognitive taxonomy of numeration systems.

The Dimensionality of Numeration Systems

1 D Systems

One of the simplest ways to represent numbers is to use stones: one stone for one, two stones for two, and so on. This Stone-Counting system only has a single dimension: the quantity of stones. The Body-Counting system used by Torres Islanders is another one dimensional system, in which the single dimension is represented by the positions of different body parts (fingers, wrist, elbow, shoulder, toes, ankles, knees, and hips). The first numeration systems invented in nearly all nations of antiquity were one dimensional systems represented by simple physical objects, such as stones, pebbles, sticks, tallies, etc. One dimensional systems are denoted as 1 D in this article.

Many 1 D systems are very efficient for small numbers. Moreover, they make a number of numerical comparison tasks simpler than with other systems (e.g., Arabic), because the size of the representation of the number is proportional to the numerical value. This means that the system has analogical properties that make comparisons simple. In addition, the operations of addition and subtraction are easy, requiring no knowledge of arithmetic properties of tables, but rather, simply the addition or removal of notational marks (see Norman, 1993, chapter 3).

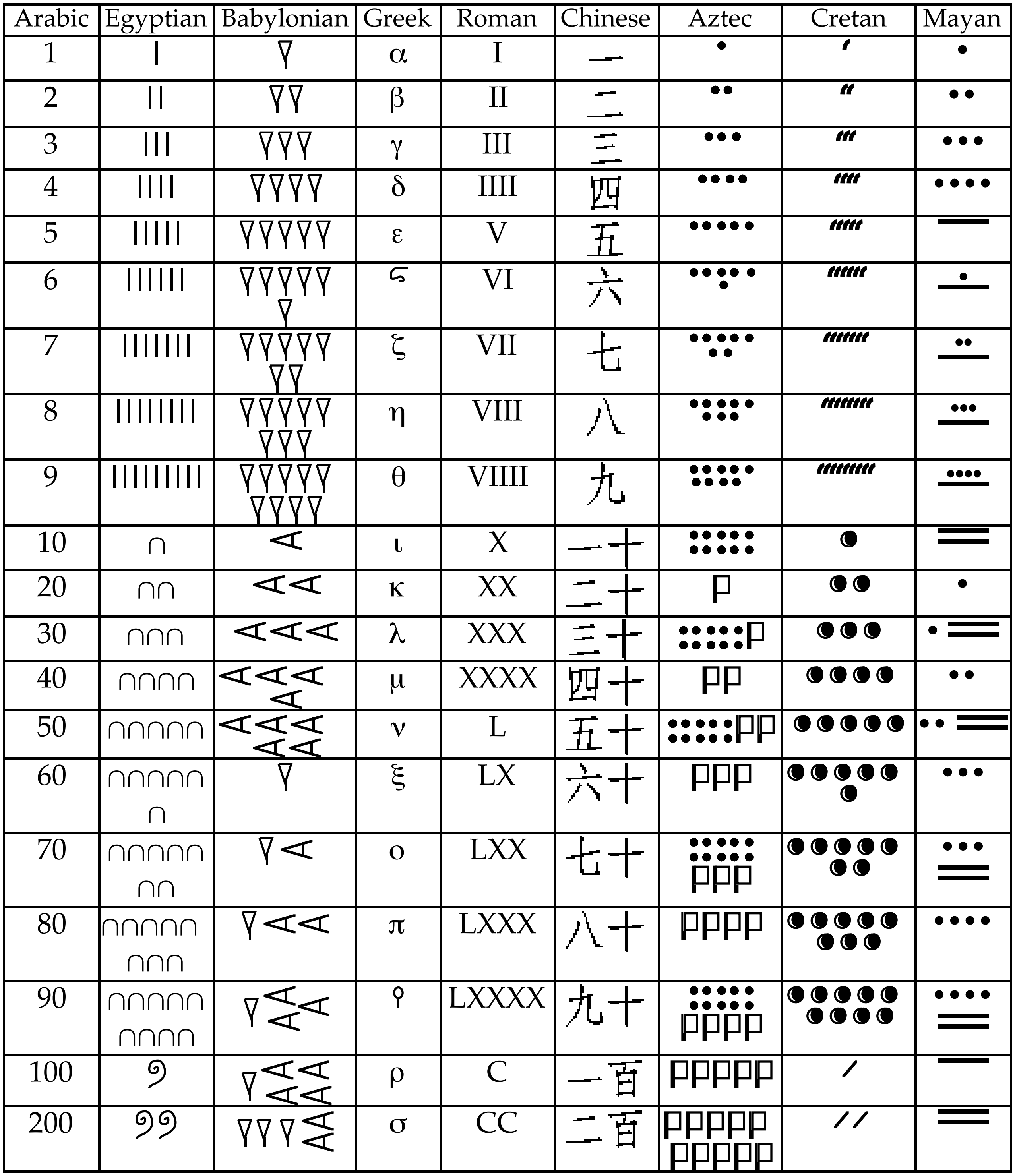

Figure 1. Examples of numeration systems. In the Roman system, the subtraction forms for four (IV, XL, etc.) and nine (IX, XC, etc.) were later inventions. We only consider the original additive forms , that is, IIII, VIIII, XXXX, LXXXX, etc.

1×1 D Systems

Although 1 D systems are efficient for small numbers, they do not work well for large numbers. For large numbers, there must be more than one dimension, for example, one *base dimension* and one *power dimension*. The power dimension decomposes a number into hierarchical groups on a *base*. The base can be raised to various powers on the power dimension. We denote two dimensional systems as 1×1 D (base×power). A number in a 1×1 D system is represented as a polynomial: $\Sigma a_i x^i$. Figure 2 shows the representational structures of six 1×1 D systems.

The two dimensions of 1×1D systems can be represented by different physical properties, which are usually shape, quantity, and position. For example, the Arabic system is a two dimensional system (Figure 2) with a base dimension represented by the shapes of the ten digits (0, 1, 2, ..., 9) and a power dimension represented by positions of the digits with a base ten. Thus, the middle 4 in 447 has a value 4 on the base dimension and a position 1 (counting from the rightmost digit, starting from zero) on the power dimension. The actual value it represents is forty (the product of its values on the base and power dimensions, i.e., 4×10^1). As another example, the base dimension of the Egyptian system (see Figure 2) is represented by quantities (the quantities of |'s, ∩'s, 𓍢's, etc.), and the power dimension is represented by shapes (|=10^0, ∩=10^1, 𓍢=10^2, etc.). Thus, 𓍢𓍢𓍢𓍢∩∩∩∩||||||| also represents four hundred and forty-seven. The details of the representations of the base and power dimensions in several other 1×1D systems are shown in Figure 2.

The two dimensions of some 1¥1 D systems are externally separable, whereas those of some others are only internally separable. In Figure 2, the base and power dimensions of all but the Greek system are externally separable. For example, we can separate the shape and position of each digit in an Arabic numeral via perceptual inspection on the physical properties of the written symbols. In this case, shape and position are two separate physical dimensions. However, the two dimensions of the Greek system (see Figure 2) are not externally separable. Its base and power dimensions are represented by a single physical dimension—shape (a one-to-two mapping): they can only be separated as two dimensions in the mind by retrieving relevant information from memory. For example, the values on the base and power dimensions of t (300) are 3 and 102, which can not be separated via perceptual inspection on the physical property of the symbol "t." The separation of the single physical dimension (shape) into a base and a power dimension in the mind is required by the Greek multiplication algorithm, which needs to process the values on the base and power dimensions separately (see Flegg, 1983; Heath, 1921).

(1×1)×1 D Systems

Some numeration systems have three dimensions: one *main power dimension*, one *sub-base dimension*, and one *sub-power dimension*. The sub-base and sub-power dimensions together form the *main base dimension*. We denote this type of three dimensional systems as (1×1)×1 D [(sub-base×sub-power)×main-power]. A numeral in a (1×1)×1D system is expressed as $\Sigma\Sigma(b_{ij}y^j)x^i$ in its abstract form. The representational struc-

Systems	Example (447)	Base	Base Dimension	Power Dimension
Abstract	$\Sigma\, a_i x^i$	x	a_i	x^i
Arabic	447 $4\times10^2+4\times10^1+7\times10^0$	10	a_i = shape 0, 1, 2, ..., 9	x^i = position ... 10^2 10^1 10^0
Egyptian	𓍢𓍢𓍢𓍢∩∩∩∩||||||| $4\times10^2+4\times10^1+7\times10^0$	10	a_i = quantity The numbers of |'s, ∩'s, 𓍢's, etc.	x^i = shape | ∩ 𓍢 ... 10^0 10^1 10^2 ...
Cretan	////◐◐◐◐ʻʻʻʻʻʻʻ $4\times10^2+4\times10^1+7\times10^0$	10	a_i = quantity The numbers of ʻ's, ◐'s, /'s, etc.	x^i = shape ʻ ◐ / ... 10^0 10^1 10^2 ...
Greek	υμζ $4\times10^2+4\times10^1+7\times10^0$	10	a_i = shape α β γ ... θ 1 2 3 ... 9	x^i = shape ι κ λ ... ϙ 1×10^1 2×10^1 3×10^1...9×10^1 ρ σ τ ... ϡ 1×10^2 2×10^2 3×10^2...9×10^2
Aztec	⚘⚑⚑●●●●●●● $1\times20^2+2\times20^1+7\times20^0$	20	a_i = quantity The numbers of ●'s, ⚑'s, ⚘'s, etc.	x^i = shape ● ⚑ ⚘ ... 20^0 20^1 20^2 ...
Chinese	四百四十七 $4\times10^2+4\times10^1+7\times10^0$	10	a_i = shape 一 二 三 ... 九 1 2 3 ... 9	x^i = shape 十 百 千 ... 10^1 10^2 10^3 ...

Figure 2. The representational structures of 1×1 D numeration systems.

Systems	Example (447)	Main Base	Sub-base	Sub-base Dimension	Sub-power Dimension	Main Power Dimension
Abstract	$\Sigma\Sigma(b_{ij}y^j)x^i$	x	y	b_{ij}	y^j	x^i
Babylonian	𒁹𒁹𒁹𒁹𒁹𒁹𒁹 𒌋𒌋𒁹𒁹𒁹𒁹𒁹𒁹𒁹 $(0\times10^1+7\times10^0)60^1$ $+(2\times10^1+7\times10^0)60^0$	60	10	b_{ij} = quantity The numbers of 𒁹's and 𒌋's	y^j = shape 𒁹 = 10^0, 𒌋 = 10^1	x^i = position ... 60^2 60^1 60^0
Mayan	• •• $\overset{\bullet\bullet}{\text{—}}$ $(0\times5^1+1\times5^0)20^2$ $+(0\times5^1+2\times5^0)20^1$ $+(1\times5^1+2\times5^0)20^0$	20	5	b_{ij} = quantity The numbers of ●'s and —'s.	y^j = shape ● = 5^0, — = 5^1	x^i = position ... 20^2 20^1 20^0
Roman	CCCCXXXXVII $(0\times5^1+4\times5^0)10^2$ $+(0\times5^1+4\times5^0)10^1$ $+(1\times5^1+2\times5^0)10^0$	10	5	b_{ij} = quantity The numbers of I's, V's, X's, L's, etc.	y^j = shape I = $10^0\times5^0$ V = $10^0\times5^1$ X = $10^1\times5^0$ L = $10^1\times5^1$ ...	x^i = shape I = $5^0\times10^0$ V = $5^0\times10^1$ X = $5^0\times10^1$ V = $5^1\times10^1$ ...

Figure 3. The representational structures of (1×1)×1 D numeration systems. In the Mayan syste. m, the second power of the main power dimension is not represented as 202, but as 20×18. This might be due to some astronomical reasons, because 20¥18 = 360, which is close to the number of days in a year.

tures of three (1×1)×1 D systems are shown in Figure 3. To illustrate this system, consider a Babylonian numeral (see Figure 3 for details):

𒐛 𒎙𒐛

$= (0\times10^1+7\times10^0)\times60^1+(2\times10^1+7\times10^0)\times60^0$

$= 7\times60^1+27\times60^0 = 420 + 27 = 447$

The main power dimension of the Babylonian system is represented by positions with a base 60. In this example, 𒎙𒐛 (the right component) is on position 0 (60^0), and 𒐛 (the left component) is on position 1 (60^1). The value of 𒐛 (the left component) on the main base dimension is 7 ($= 0\times10^1+7\times10^0$). The actual value it represents is the product of its values on the main base and main power dimensions: $(0\times10^1+7\times10^0)\times60^1 = 420$. The main base dimension is composed of a sub-base dimension represented by quantity (the quantities of 𒁹's and 𒌋's) and a sub-power dimension represented by shape (𒁹 = 10^0 and 𒌋 = 10^1) with a base 10. For example, 𒎙𒐛 can be decomposed as $2\times10^1+7\times10^0$.

Similar to 1×1 D systems, the three dimensions in a (1×1)×1 D system can be externally separable or only internally separable. For example, the sub-base, sub-power, and the main power dimensions in the Babylonian and the Mayan systems are externally separable with each other. In the Roman system, although the sub-base dimension is externally separable from the sub-power and the main power dimensions, the sub-power and the main power dimensions, which are represented by a single physical dimension (shape), are only internally separable. For example, L (50) has a value 5^1 on the sub-power dimension and a value 10^1 on the main power dimension, which can only be separated in the mind.

Hierarchical Structure and Cognitive Taxonomy

Based on the analysis of the dimensionality of numeration systems, we can analyze number representations at four levels: dimensionality, dimensional representation, bases, and symbol representation.

Each level has an *abstract structure* that can be implemented in different ways. The different representations at each level are isomorphic to each other in the sense that they all have the same abstract structure at that particular level (Figure 4).

At the *level of dimensionality*, different numeration systems can have different dimensionalities: 1D, 1×1D, (1×1)×1D, and others. However, they are all isomorphic to each other at this level in the sense that they all represent the same entities—numbers. This level mainly affects the efficiency of information encoding. 1 D systems are linear, while 1×1 D and (1×1)×1 D systems are polynomial. Polynomial systems encode information more efficiently than linear systems: the number of symbols needed to encode a number in a polynomial system is proportional to the logarithm of the number of symbols needed to encode the same number in a linear system.

At the *level of dimensional representations*, isomorphic numeration systems have the same dimensionality but different dimensional representations. The physical properties used to represent the dimensions of numeration systems are usually quantity (Q), position (P), and shape (S). For example, the base and power dimensions of 1×1D systems can be represented by shape and position (S×P, Arabic system), shape and shape

(S×S, Chinese system), quantity and shape (Q×S, Egyptian system), etc. This level is crucial for the representational effect of numeration systems. The second part of this paper is devoted for the analysis of the representational properties at this level.

At the *level of bases*, isomorphic numeration systems have the same dimensionality, same dimensional representations, but different bases. For example, both the Egyptian and the Aztec systems are 1×1D systems, and the base and power dimensions of both systems are represented by quantity and shape. However, the base of the Egyptian system is ten while that of the Aztec system is twenty (see Figure 2). This level is important for tasks involving addition and manipulation tables: the larger a base is, the larger the addition and multiplication tables are and the harder they can be memorized and retrieved.

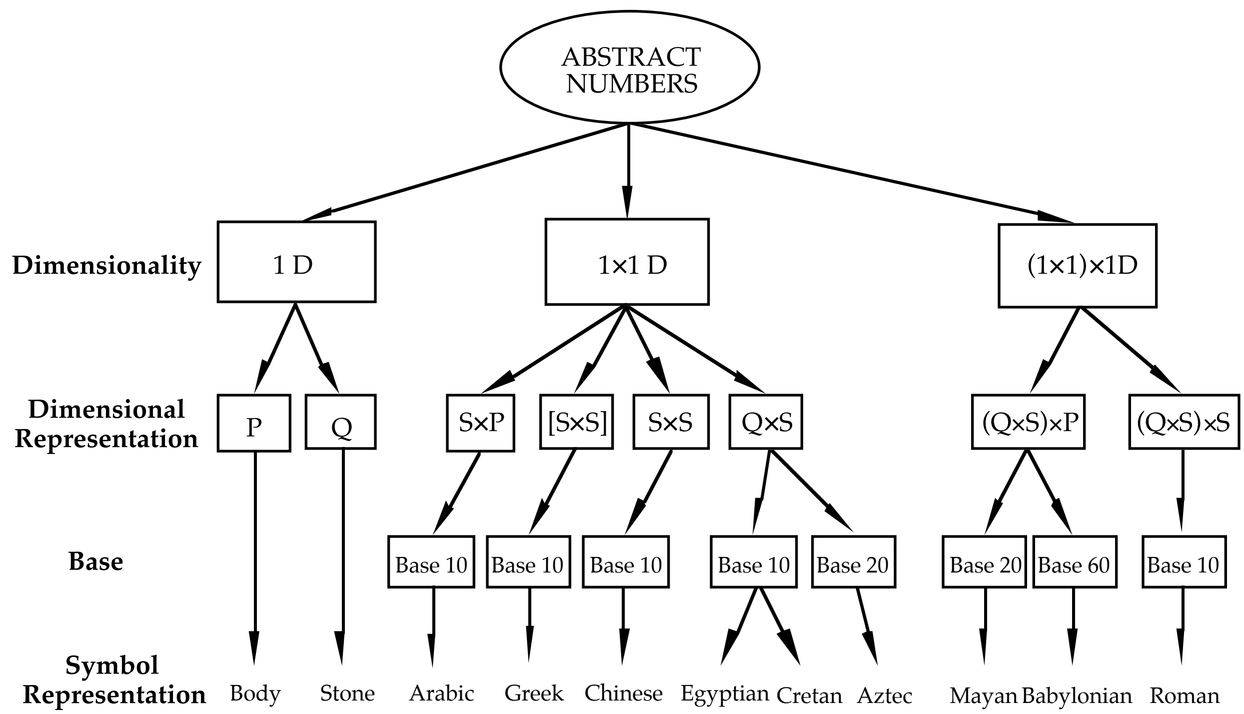

Figure 4. The hierarchical structure of number representations. This is a also a cognitive taxonomy of numeration systems. At the level of dimensionality, different systems have different dimensionalities. At the level of dimensional representations, the dimensions of different systems are represented by different physical properties. P = Position, Q = Quantity, S = Shape. The two dimensions of the Greek system ([S¥S]) are represented in the mind and only separable in the mind. At the level of bases, different systems may have different bases. At the level of symbol representations, different systems use different symbols.

At the *level of symbol representations*, isomorphic numeration systems have the same abstract structures at the previous three levels. However, different symbols are used. For example, both the Egyptian and the Cretan systems are 1×1D systems, the two dimensions of both systems are represented by quantity and shape, and both systems have the base ten. However, in the Egyptian system, the symbols for 10^0, 10^1, and 10^2 are |, ∩, and 𓍢, whereas in the Cretan system, the corresponding symbols are ′, ●, and ⁄. This level mainly affects the reading and writing of individual symbols.

The hierarchical structure of number representations in Figure 4 is in fact a cognitive taxonomy of numeration systems. For example, the Egyptian and Cretan systems are in the same group at the level of symbol representations; the Mayan and Babylonian systems are in the same group at the level of bases; the Arabic, Greek, Chinese, Egyptian, Cretan, and Aztec systems are in the same group at the level of dimensional representations; and all the systems in Figure 4 are in the same group at the level of dimensionality. Under this taxonomy, the lower the level at which two systems are in the same group, the more similar they are. For example, the Egyptian and the Cretan systems are more similar to each other than the Arabic and the Babylonian systems, because the former two are in the same group at the level of symbol representations whereas the latter two at the level of dimensionality.

Although this cognitive taxonomy was derived from the eleven systems in Figures 2 and 3, it can classify nearly all numeration systems that have been invented across the world. Let us consider a few more systems (see Ifrah, 1987, for detailed descriptions of these systems). At the level of symbol representations, the Hebrew alphabetic system is in the same group as the Greek system, and the Greek acrophonic, Dalmatian, and Etruscan systems are in the same group as the Roman system. At the level of bases, the Chinese scientific system is in the same group as the Mayan system.

In addition to numeration systems of written numerals, this taxonomy can also classify numeration systems of object numerals. The following are a few examples (see Ifrah, 1987), in which P = Position, Q = Quantity, and S = Shape. The Peruvian knotted string system is a P×Q (base 10) system; the Chimpu (knotted strings used by the Indians of Peru and Bolivia) is a Q×Q (base 10) system; the knotted string system used by the German millers is a S×S (base 10) system; the Roman counting board, the Chinese abacus, and the Japanese Soroban are (Q×P)×P (main base 10 and sub-base 5) systems, and the Russian abacus is a Q×P (base 10) system.

The hierarchical structures of the numeration systems discussed above are based on the representations of small numbers (usually less than 1000). For large numbers, the representational structures of some systems may have different forms. The change is mainly due to the representations of power dimensions. The representational structures of numeration systems with power dimensions represented by positions usually do not change, because positions can be extended indefinitely. However, for those systems with power dimensions represented by shapes, their representational structures sometimes have to be changed to represent large numbers, because practically, shapes can not be extended indefinitely. For example, the Roman system is a (Q×S)×S system for numbers up to one thousand. For numbers above one thousand, one version of the Roman system becomes a ((Q×S)×S)×S system. A new power dimension represented by shape is added: a horizontal bar æ is used to represent 103, and an open frame ⌈⌉ for 105. For example, $\overline{\text{CCXXXVIII}}$ represents 238, 000, and ⌈DCCCXXVII⌉ represents 82, 700, 000.

THE DISTRIBUTED REPRESENTATION OF NUMBERS

The Theory of Distributed Representations

In complex numerical tasks, as well as in many other cognitive tasks, people need to process the information perceived from external representations and the information retrieved from internal representations in an interwoven, integrative, and dynamic manner. External representations are the representations in the environment, as physical symbols or objects (e.g., written symbols, beads of abacuses, etc.) and external rules, constraints, or relations embedded in physical configurations (e.g., spatial relations of written digits, visual and spatial layouts of diagrams, physical constraints in abacuses, etc.). The information in external representations can be picked up by perceptual processes. In contrast, internal representations are the representations in the mind, as propositions, productions, schemas, mental images, neural networks, or other forms. The information in internal representations has to be retrieved from memory by cognitive processes. For example, in the task of 735 × 278 with paper and pencil, the internal representations are the values of individual symbols (e.g., the value of the arbitrary symbol "7" is seven), the addition and multiplication tables, arithmetic procedures, etc., which have to be retrieved from memory; and the external representations are the shapes and positions of the symbols, the spatial relations of partial products, etc., which can be perceptually inspected from the environment.

Zhang & Norman (1994a; Zhang, 1992, 1995) developed a theory of distributed representations to account for the behavior in *distributed cognitive tasks*—tasks that involve both internal and external representations. In this view, the representation of a distributed cognitive task is neither solely internal nor solely external, but distributed as a system of distributed representations with internal and external representations as two indispensable parts. Thus, external representations are intrinsic components and essential ingredients of distributed cognitive tasks. They need not be re-represented as internal representations in order to be involved in distributed cognitive tasks: they can directly activate perceptual processes and directly provide perceptual information that, in conjunction with internal representations, determine people's behavior. External representations have rich structures. Without a means of accommodating external representations in its own right, we sometimes have to postulate complex internal representations to account for the structure of behavior, much of which, however, is merely a reflection of the structure of external representations.

In next section we apply the principles of distributed representations to analyze the representation of information in the basic structure of numeration systems—dimensions. In the later part for multiplication, we will further analyze how the information needed to carry out multiplication is distributed across internal and external representations.

The Distributed Representation of Dimensions

Dimensions are the basic structures of numeration systems. Thus, it is important to understand how dimensions are represented in numeration systems.

Psychological Scales

Every dimension is on a certain type of scale, which is the abstract measurement property of the dimension. Stevens (1946) identified four major types of psychological scales: ratio, interval, ordinal, and nominal. Each type has one or more of the following formal properties: category, magnitude, equal interval, and absolute zero. *Category* refers to the property that the instances on a scale can be distinguished from each another. *Magnitude* refers to the property that one instance on a scale can be judged greater than, less than, or equal to another instance on the same scale. *Equal interval* refers to the property that the magnitude of an instance represented by a unit on the scale is the same regardless of where on the scale the unit falls. An *absolute zero* is a value which indicates that nothing at all of the property being represented exists.

Nominal scales only have one formal property: category. Names of people are an example of nominal scales: they only discriminate different entities but have no information about magnitudes, intervals, and ratios. Ordinal scales have two formal properties: category and magnitude. The ranking of movie quality is an example of ordinal scales: a movie ranked "1" is better than a movie ranked "2" (magnitude) and the quality of a movie ranked "5" is different from that of a movie ranked "7" (category). However, the rankings themselves tell us nothing about the differences and ratios between the rankings. Interval scales have three formal properties: category, magnitude, and equal interval. Time is an example of interval scales: 02:00 is different from 22:00 (category), 14:00 is later than 09:00 (magnitude), and the difference between 15:00 and 14:00 is the same as between 09:00 and 08:00 (equal interval). However, time does not have an absolute zero (in a realistic sense). Thus, we cannot say that 10:00 is twice as late as 05:00. Ratio scales have all of the four formal properties: category, magnitude, equal interval, and absolute zero. Length is an example of ratio scales: 1 inch is different from 3 inches (category), 10 inches are longer than 5 inches (magnitude), the difference between 10 and 11 inches is the same as the difference between 100 and 101 inches (equal interval), and 0 inch means the nonexistence of length (absolute zero). For length, we can say that 10 inches are twice as long as 5 inches.

Distributed Representation of Scale Information

The base and power dimensions of all numeration systems are abstract dimensions with ratio scales. In different numeration systems, these abstract ratio dimensions are implemented by different physical dimensions with their own scale types, which are usually quantity (ratio), shape (nominal), and position (ratio). Figure 5 shows the representation of base dimension in the Arabic and Egyptian systems. In Figure 5, the abstract information space for base dimension contains the four formal properties of ratio scales, since base dimension is on a ratio scale. In the Arabic system, the base dimension is represented by shape—an external physical dimension with a nominal scale. Because a nominal dimension only has category information, shape can only represent the category information of the ratio base dimension in external representations: the other three types of information have to be represented in internal representations, that is, memorized in the mind (Figure 5A). Thus, the representation of base dimension in the Arabic system is a distributed representation with category information represented externally by shape and other three types of information represented internally in memory. In the Egyptian system, the base dimension is represented by quantity—an external physical dimension with a ratio scale. Thus, in this case, all four types of information of the ratio base dimension are represented externally (Figure 5B). Similar to quantity, position is also a ratio dimension. Therefore, if the base or power dimension of a numeration system is represented by position, its scale information is all represented externally.

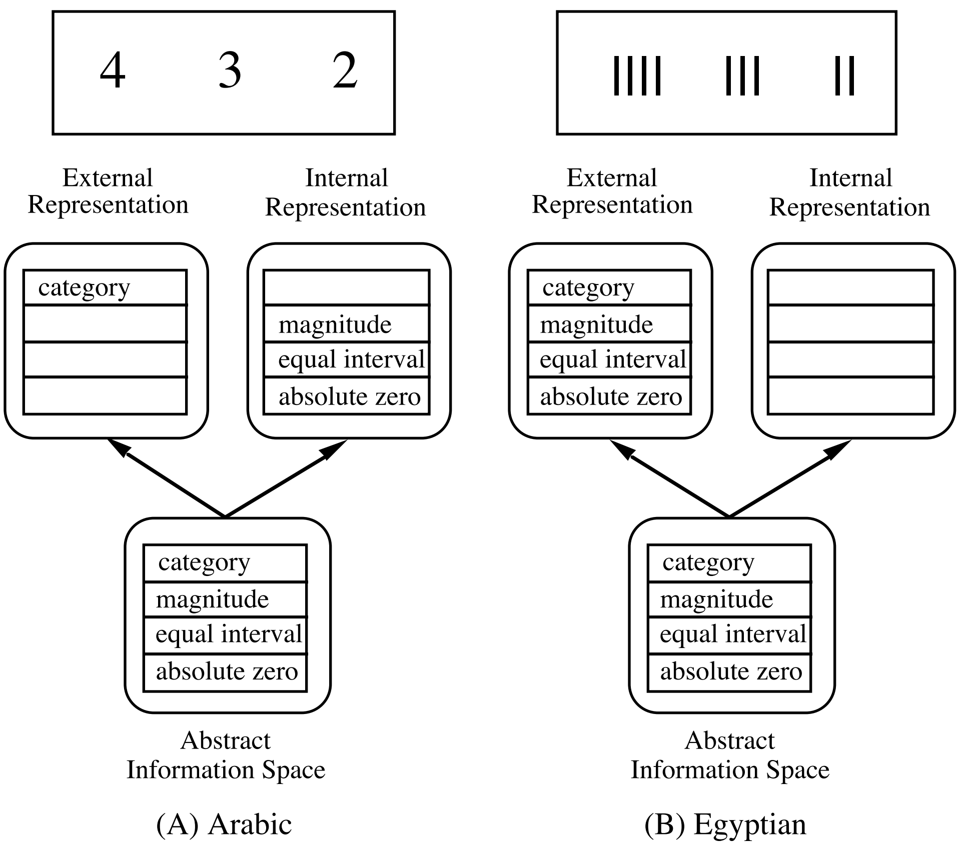

Figure 5. The distributed representation of dimensions. The scale information of a dimension is in the abstract information space, which is distributed across an internal and an external representation. (A) The representation of base dimension in the Arabic system. The category information of ratio base dimension is represented externally by shape (a nominal dimension) but the other three types of information are represented internally in memory. (B) The representation of base dimension in the Egyptian system. All four types of information of ratio base dimension are represented externally by quantity (also a ratio dimension).

The information in external representations can be picked up by perceptual processes, whereas the information in internal representations has to be retrieved from memory. For example, in Figure 5A for the Arabic system, the three symbols "4", "3", and "2" can be perceptually identified as three different entities (category information). However, the information about whether "4" is smaller or larger than "3" (magnitude), whether the difference between "4" and "3" is the same as or different from that between "3" and "2" (equal interval), and whether "4" is twice as large as "2" (absolute zero) has to be retrieved from memory. In contrast, in Figure 5B for the Egyptian system, all four types of information can be picked up by perceptual processes: ||||, |||, and || are different entities (category), |||| is larger than ||| (magnitude), the difference between |||| and ||| is the same as the difference between ||| and || (equal interval), and |||| is twice as large as || (absolute zero).

The scale information of a dimension, however, is only relational information: it does not specify the absolute values on the dimension. Thus, the representation of the absolute values on a dimension needs to be considered separately. The absolute values on base and power dimensions (base values and power values) are represented internally by shapes but externally by positions and quantities. For example, even if "4", "3", and "2" can be perceptually identified as three different entities, the absolute values these symbols represent have to be retrieved from memory. In contrast, the absolute values represented by "||||", "|||", and "||" are external and can be picked up by perceptual processes.

Magnitude Comparison

Let us use magnitude comparison as an example to illustrate how the above analysis of dimensional representations can be applied to simple numerical tasks. Comparing the relative magnitudes of numbers only requires the magnitude information of the numbers. For a 1 D system, if the dimension is represented by an ordinal or a higher dimension, the magnitude information needed for comparison is external. For example, the quantity dimension in the Stone-Counting system is an external representation of magnitude information. For a 1×1 D system, if both the base and power dimensions are on ordinal or higher scales (e.g., Russian abacus), the magnitude information needed for comparison is external, and if both dimensions are on nominal scales (e.g., Greek system), the information is internal. Otherwise, it depends on the representation of each dimension. For example, for the Arabic system (a Position×Shape system, ratio power dimension and nominal base dimension), the relative magnitude information of two numbers is external if the two numbers have different highest power values (e.g., 75 vs. 436) and internal if they have the same highest power values (e.g., 75 vs. 43).

The Interaction between Dimensions

Because most numeration systems have more than one dimension, it is also important to understand how the dimensions interact with each other. The dimensions of a multi-dimensional stimulus can be either separable or integral. Separable dimensions are those whose component dimensions can be directly and automatically separated and perceived. Integral dimensions can only be perceived in a holistic fashion: they can not be separated without a secondary process that is not automatically executed (Garner, 1974).

Although the concept of separability usually refers to external physical dimensions, we use it to refer to internal cognitive dimensions as well. If the information needed to separate two dimensions can be picked up by perceptual processes from external representations, we say that these two dimensions are externally separable. For example, the shape (base dimension) and position (power dimension) of the Arabic system are two physical dimensions that are externally separable via perceptual processes. If the information needed to separate two dimensions is not available in external representations and can only be retrieved from internal representations, we say that these two dimensions are externally inseparable and only internally separable. In the Greek system, for example, the base and power dimensions are represented by a single physical dimension (shape). They can only be separated internally.

Since many numerical tasks (e.g., multiplication and addition under polynomial representations) need to process the base and power dimensions separately, whether the dimensions of a numeration system are externally separable or not can dramatically affect task difficulties. We will see this effect in next section.

A NUMERICAL TASK: MULTIPLICATION

In this section, we apply the principles of distributed representations to analyze how representational formats affect complex numerical tasks. To make our demonstration simple and clear, we choose multiplication on the task side and 1×1 D numeration systems on the representation side. Since numeration systems are represented at four levels, the representational properties at different levels can affect numerical tasks in different ways. Here we only focus on the representational properties at the level of dimensional representations, since whether the dimensions of numeration systems are represented externally and whether they are externally separable can dramatically affect the difficulty of a numerical task.

The Hierarchical Structure of Multiplication

Multiplication can be analyzed at three levels (Figure 6): algebraic structures, algorithms, and number representations. At the level of algebraic structures, different multiplication methods have different algebraic structures. The algebraic structures of three typical methods, simple addition, binary, and polynomial methods, are as follows:

Simple Addition:

$$N_1 \times N_2 = N_1 + N_1 + N_1 + \ldots + N_1 \quad \text{(add } N_2 \text{ times)}$$

Binary:

$$N_1 \times N_2 = N_1 \times \sum a_i 2^i = \sum N_1 a_i 2^i$$

Polynomial:

$$N_1 \times N_2 = \sum a_i x^i \times \sum a_j x^j = \sum\sum a_i x^i a_j x^j = \sum\sum a_i a_j x^{i+j}$$

The simple addition method is simply adding the multiplicand the number of times indicated by the multiplier. For the doubling method, multiplication is performed by (a) decomposing the multiplier into its binary representation, (b) multiplying the multiplicand with each term of the binary representation of the multiplier, and (c) adding the partial products. Because the terms of the binary representation are either in the form of 1×2^n or 0×2^n, the multiplication of the multiplicand with each term of the binary representation of the multiplier can be performed by successively doubling the multiplicand. For the polynomial method, multiplication is performed by multiplying every term of the multiplicand with every term of the multiplier and then adding the partial products together

At the level of algorithms, different algorithms can be applied to the same algebraic structure. For example, the binary method can be realized by the Egyptian algorithm and the Russian algorithm (for details of these two algorithms, see Zhang, 1992), and the polynomial method can be realized by the Standard algorithm and the Greek algorithm, which are described in a later section. At the level of number representations, multiplication can be performed under different numeration systems.

Here we only analyze the polynomial method, the one that includes the Standard algorithm we are using today. The simple addition and the binary methods, which can be reduced to addition, were analyzed in Zhang (1992).

The Polynomial Method of Multiplication

A numeral in a 1×1 D system is represented as a polynomial: $\sum a_i x^i$. Multiplication by the polynomial method is performed by multiplying every term of the multiplicand with every term of the multiplier and then adding the partial products together, regardless of which particular algorithm (e.g., the standard or the Greek algorithm, see next section) is used. Thus, the two basic components of polynomial multiplication are the multiplication of individual terms and the addition of partial products. Here we only show how representational formats affect the processes of term multiplication. (A detailed analysis involving both term multiplication and partial product addition can be found in Zhang, 1992.)

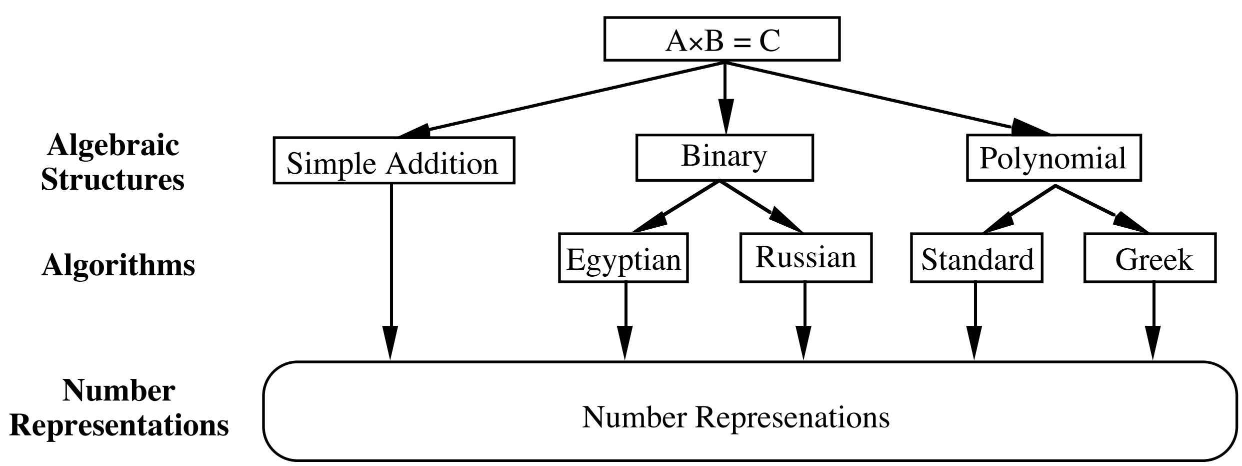

Figure 6. The hierarchical structure of multiplication.

For all 1×1 D systems, term multiplication ($a_ix^i \times b_jy^j$) has the same set of six basic steps (see Figure 7): one step (Step 1) to separate the power and base dimensions, two steps (Steps 2a and 2b) to multiply the base values, two steps (Steps 3a and 3b) to add the power values, and one step (Step 4) to combine the power and base values of the final product. Although the six steps need not be followed in the exact order, there are certain constraints: Step 1 must be the first step and Step 4 must be the last step, but Steps 2a and 2b can either proceed or follow Steps 3a and 3b. The information needed for each step can be either perceived from external representations or retrieved from internal representations. Figure 7 and the following descriptions show the analyses of the six steps for the Arabic, Greek, and Egyptian systems.

Step	**Abstract:** $a_ix^i \times b_jx^j$	**Greek:** λ×δ	**Egyptian:** ∩∩∩×||||	**Arabic:** 30×4
1	separate power & base dimensions	**internal**: internally separable	**external**: externally separable	**external**: externally separable
2a	get base values of a_ix^i & b_jx^j	**internal**: a_i, b_j = shapes	**external**: a_i, b_j = quantities	**internal**: a_i, b_j = shapes
	$B(a_ix^i) = a_i$ $B(b_jx^j) = b_j$	B(λ) = γ B(δ) = δ	B(∩∩∩) = ||| B(||||) = ||||	B(30) = 3 B(4) = 4
2b	multiply base values	**internal**: multiplication table	**internal**: multiplication table	**internal**: multiplication table
	$a_i \times b_j = c_{ij}$	γ × δ = ιβ	|||×|||| = ∩||	3×4 = 12
3a	get power values of a_ix^i & b_jx^j	**internal**: i, j = shapes	**internal**: i, j = shapes	**external**: i, j = positions
	$P(a_ix^i) = i$ $P(b_jx^j) = j$	P(λ) = 1 P(δ) = 0	P(∩∩∩) = 1 P(||||) = 0	P(30) = 1 P(4) = 0
3b	add power values	**internal**: addition table	**internal**: addition table	**external**: positions
	$P(a_ix^ib_jx^j)$ $= P(a_ix^i)+P(b_jx^j)$ $= i+j = p_{ij}$	P(λ×δ) = P(λ)+P(δ) = 1+0 = 1	P(∩∩∩×||||) = P(∩∩∩)+P(||||) = 1+0 = 1	P(30×4) = P(30)+P(4) = 1+0 = 1
4	attach power values	**internal**: shapes	**internal**: shapes	**external**: positions
	$a_ix^i \times b_jx^j$ $= c_{ij} \times x^{p_{ij}}$	λ×δ = (ιβ)×10^1 = ρκ	∩∩∩×|||| = (∩||)×10^1 = 𓍢 ∩∩	30×4 = 12×10^1 = 120

Figure 7. The six basic steps of term multiplication for the polynomial method. See text for details.

Step 1: Separate power and base dimensions. The power and base dimensions are externally separable for the Arabic system (position and shape) and the Egyptian system (shape and quantity) but only internally separable for the Greek system (a single shape dimension for both power and base dimensions). Thus, the information needed for this step is external for the Arabic and Egyptian systems but internal for the Greek system.

Step 2a: Get base values of $a_i x^i$ and $b_j y^j$: a_i , b_j. The base values are represented internally in the Greek system (shape) and the Arabic system (shape) but externally in the Egyptian system (quantity). Thus the information needed for this step is internal for the Greek and Arabic systems but external for the Egyptian system.

Step 2b: Multiply base values: $a_i \times b_j = c_{ij}$. The information needed for this step is internal for all three systems, which need to be retrieved from an internal multiplication table[2].

Step 3a: Get power values of $a_i x^i$ and $b_j y^j$: i, j. The power values are represented externally in the Arabic system (position) but internally in the Greek system (shape) and the Egyptian system (shape). Thus, the information needed for this step is external for the Arabic system but internal for the Greek and Egyptian systems.

Step 3b: Add power values: $i + j = p_{ij}$. In the Arabic system, since adding the power values of two terms can be performed perceptually by shifting positions, the information needed for this step is external. In the Greek and Egyptian systems, since the sums of power values have to be retrieved internally from the mental addition table, the information needed for this step is internal.

Step 4: Attach power values to the product of base values: $a_i x^i \times b_j y^j = c_{ij} \times x^{p_{ij}}$. The sum of power values of the two terms need to be attached to the product of the base values of the two terms to get the final result. The information needed for this step is external for the Arabic system because the sum of power values can be attached to the product of base values by shifting positions. For the Greek and Egyptian systems, the information needed to this step is internal because power values in both systems are represented internally by shapes.

From the above analysis, we see that the information needed for the six basic steps of term multiplication is distributed across internal and external representations. Figure 8 shows the distributed representations of term multiplication for the Greek, Egyptian, and Arabic systems. If we assume that with all other conditions kept the same, the more information needs to be retrieved from internal representations, the harder the task (e.g., due to working memory load), then for term multiplication, the Greek system (six internal steps) is harder than the Egyptian system (four internal steps), which in turn is harder than the Arabic system (two internal steps)[3]. We need to note that difficulty

[2] The multiplication table can be externalized. For example, Napier's Bone is an external representation of the multiplication table (see Zhang, 1992).

[3] A complete multiplication task involves not just the multiplication of individual terms but also the addition of partial products. When the addition of partial products is considered, the same difficulty order remains: Greek > Egyptian > Arabic (see Zhang, 1992).

can be measured along many dimensions, some of which may have trade-off between each other. The amount of internal and external information is only one of the difficulty measures.

Multiplication Algorithms for the Polynomial Method

The polynomial method of multiplication can be realized by different algorithms. Although different algorithms have the same set of six basic steps of term multiplication, they usually have different orders for the multiplication of individual terms and different groupings for the partial products. Figure 9 compares two algorithms: the Greek algorithm used in ancient times by Greeks and the Standard algorithm used today by people in nearly all cultures.

The Standard algorithm starts the multiplication of individual terms from the lowest term of the multiplier, while the Greek algorithm starts from the highest term of the multiplicand. The Standard algorithm groups the partial products that have common powers by their spatial relations (positions), which makes the addition of partial products easier. The Greek algorithm, however, does not group partial products that have common powers and does not make use of spatial relations for the addition of partial products.

Under the same algorithm, the Arabic system is more efficient than the Greek system because the former is more efficient than the latter for term multiplication. Under the same numeration system, the Standard algorithm is more efficient than the Greek algorithm because the spatial grouping of partial products in the Standard algorithm makes the addition of partial products easier.

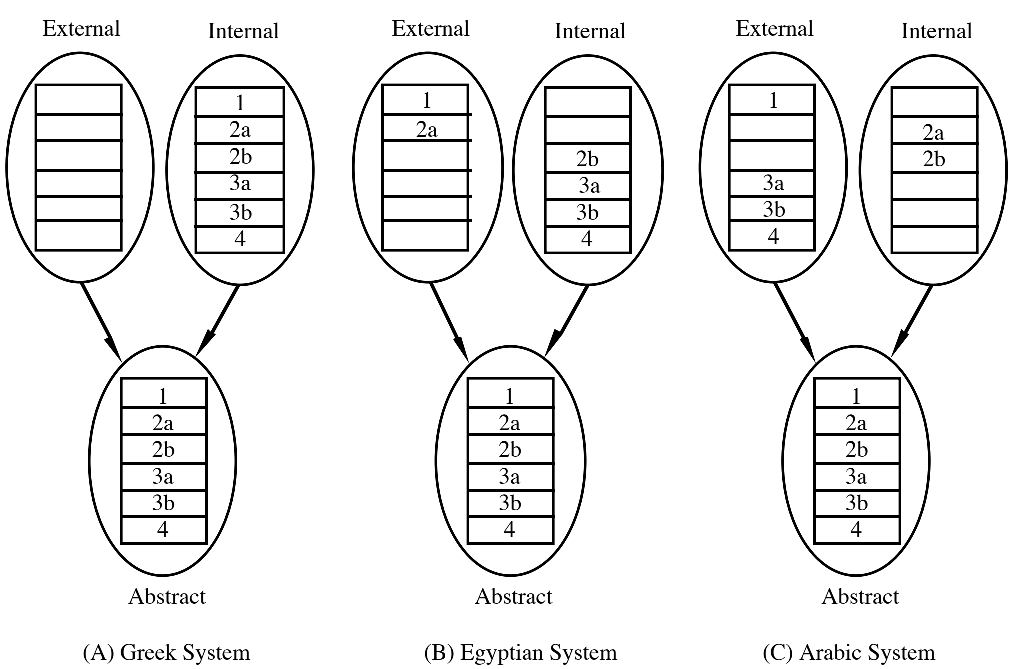

Figure 8. The distributed representation of the information needed for the six basic steps of term multiplication.

Procedures	Abstract Form	Greek system	Arabic system
Greek Algorithm	$a_1x^1 \quad a_0x^0$	ι ζ	10 7
	$b_1x^1 \quad b_0x^0$	ι γ	10 3
	$a_1b_1x^2 \quad a_1b_0x^1$	ρ λ	100 30
	$a_0b_1x^1 \quad a_0b_0x^0$	ο κα	70 21
	$(a_1b_1x^2+a_0b_1x^1) + (a_1b_0x^1+a_0b_0x^0)$	ρο να σκα	170 51 221
Standard Algorithm	$a_1x^1 \quad a_0x^0$	ι ζ	1 7
	$b_1x^1 \quad b_0x^0$	ι γ	1 3
	$a_0b_0x^0$	κ α	2 1
	$a_1b_0x^1$	λ	3
	$a_0b_1x^1$	ο	7
	$a_1b_1x^2$	ρ	1
	$a_1b_1x^2 + (a_1b_0x^1+a_0b_1x^1) + a_0b_0x^0$	σ κ α	2 2 1

Figure 9. The Greek and the Standard algorithms of multiplication under the Greek and Arabic systems. The example is 17×13.

GENERAL DISCUSSION

In this article, we have explored the cognitive aspects of numeration systems, focusing on external number representations and their interaction with internal representations. We show that the information that needs to be processed in complex numerical tasks is distributed across internal and external representations.

We analyzed numeration systems at four levels: dimensionality, dimensional representations, bases, and symbol representations. The representational properties at these levels can affect the processes of numerical tasks in different ways and are responsible for different aspects of the representational effect in numerical tasks. These levels provide a cognitive taxonomy that can classify most numeration systems that have been invented across the world. We suggest that the hierarchical structure can be considered as a theoretical framework for systematic studies of number representations. Among the four levels, the level of dimensional representations is most important: whether the dimensions and their absolute values are represented externally or internally and whether the dimensions are externally or internally separable can dramatically affect the difficulty of a numerical task. Thus, when we analyze the polynomial multiplication method for three 1×1 D systems (Arabic, Egyptian, and Greek) using the amount of information that has to be retrieved from internal representations as a measure of problem difficulty, the Greek system is found to be the hardest among the three systems and the Arabic system the easiest.

A Cognitive Model of Distributed Numerical Cognition

Any distributed cognitive task can be analyzed into three aspects: formal structures, representations, and processes. Formal structures specify the information that has to be processed in a task; representations specify how the information to be processed is represented across internal and external representations; and processes specify the actual mechanisms of information processing. These three aspects are closely interrelated: the same formal structure can be implemented by different representations, and different representations can activate different processes. Any study of distributed cognitive tasks has to consider all of these three aspects. Our present study of numeration systems and numerical tasks, however, only focused on their formal structures and representations. In this section, we use the term multiplication of Arabic numerals as a special case to outline a cognitive model of distributed numerical cognition with several assumptions about the processing mechanisms.

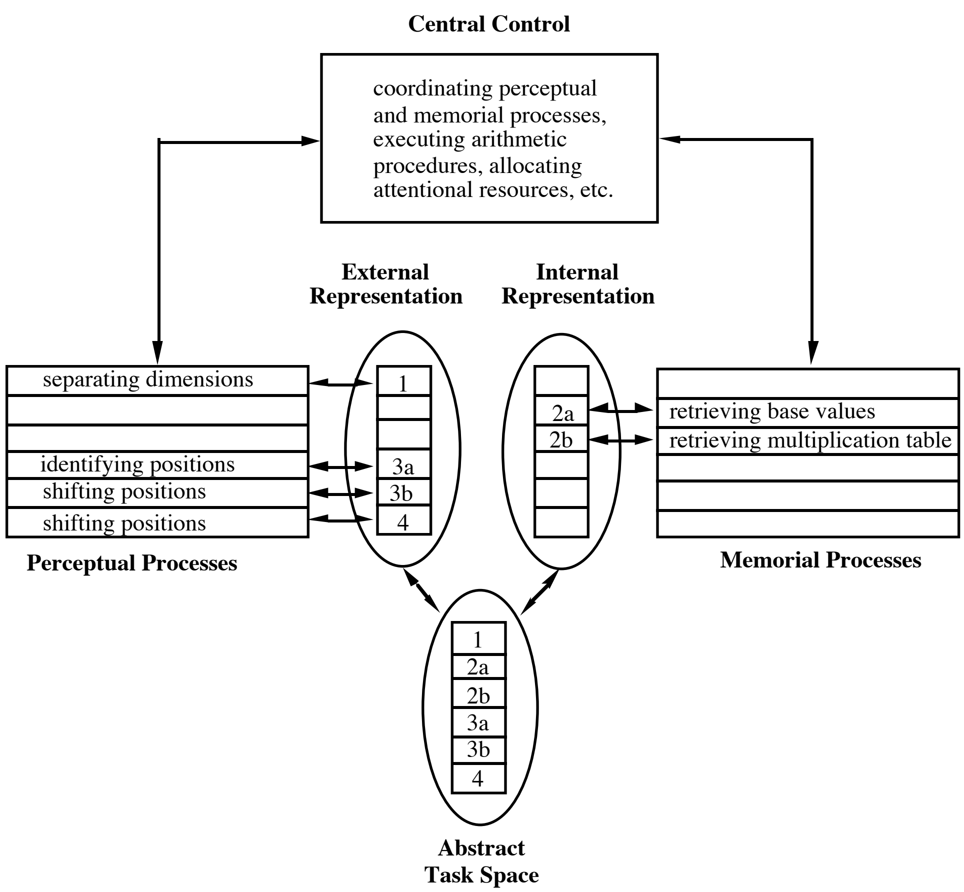

Figure 10. The model of distributed numerical cognition. See text for details.

The model is shown in Figure 10. The abstract task space specifies the formal structure of the task—the six basic steps of term multiplication, and the information that has to be processed—the information needed for the six basic steps of term multiplication. The information needed for the six basic steps is distributed across internal and external representations. For Arabic numerals, the information for Steps 1, 3a, 3b, and 4 is represented externally in the environment and the information for Steps 2a and 2b is represented internally in memory (see also Figures 7 and 8). External information is picked up from external representations by perceptual processes. For Arabic numerals, these perceptual processes include separating the base and power dimensions (Step 1), identifying the positions to get power values (Step 3a), shifting positions to add power values (Step 3b), and shifting positions to combine the base and power values of the final product (Step 4). Internal information is retrieved from internal representations by memorial processes. For Arabic numerals, these memorial processes include retrieving the base values from memory (Step 2a) and retrieving multiplication facts from the multiplication table in memory (Step 2b). The central control coordinates the perceptual and memorial processes, executes arithmetic procedures, allocates attentional resources, and performs other processes that are necessary for the completion of the task.

The first assumption of our model is for representations: the information needed for a numerical task is distributed across internal and external representations. It has been shown (Zhang & Norman, 1994a; Zhang, 1995) that manipulating the information distributed across internal and external representations can dramatically affect the difficulty of a task. The second assumption is for processes: external representations activate perceptual processes and internal representations activate memorial processes. Since perceptual processes can be direct, automatic, unconscious, and efficient, their involvement can reduce the difficulty of a task. The third assumption is for the central control, whose major function is the coordination of perceptual and memorial processes. Although there have been several studies on the interplay between perception and memory in terms of attention switching (e.g., Carlson, Wenger, & Sullivan, 1993; Dark, 1990; Weber, Byrd, & Noll, 1986), it is still unclear how perceptual and memorial processes interact with each other. For instance, the reason we did not mention working memory in the central control is because it is not clear to us how the framework of working memory (Baddeley, 1986; Baddeley & Hitch, 1974) can fit here. Can perception, especially the automatic and direct kind, bypass working memory to directly participate in complex distributed cognitive tasks? In addition to the internal working memory, is there a separate external working memory? If yes, what are its functions and what is its relation to the internal working memory? These questions are central to the understanding of the nature of distributed cognitive tasks. They deserve systematic studies, both empirically and theoretically.

Is the Arabic System Special?

Very few things in the world are as universal as the Arabic numeration system.

It has often been argued that the place-value notation (position) of the Arabic system is one of the most ingenious inventions of the human mind and it is what makes the Arabic system so special. Our analysis of the representational properties of numeration systems allows us to examine the question of why the Arabic numeration system is so special.

Based on our analysis, the place-value notation is not sufficient to make the Arabic system so special, since positions were also used to represent the power dimensions in many other systems, including the Babylonian, Chinese scientific, and Mayan written numeration systems, and the abacus, counting board, Peruvian knotted string, and other object numeration systems. Then, what properties of the Arabic system make it so special? Let us first consider the properties of the Arabic system at its four levels of representations.

At the level of dimensionality, the Arabic system is a 1×1 D system, which is more efficient for information encoding than 1 D systems. Although (1×1)×1 D systems (e.g., Babylonian, Mayan, Roman, etc.) are as efficient as 1×1 D systems for information encoding, the extra third dimension only introduces unnecessary complexity.

At the level of dimensional representations, the base and power dimensions of the Arabic system are externally separable, and the power dimension and its power values are represented externally by positions. Positions can handle large numbers efficiently. However, the base values and three of the four types of scale information of the base dimension are represented internally by shapes. If we are only concerned with whether the base and power dimensions and their values are represented internally or externally, the Russian abacus, whose base dimension (quantity) as well as power dimension (position) are both represented externally, are superior. In fact, abacuses are always superior to written numerals for simple calculation such as addition and subtraction. But if we consider other factors such as ease of reading and writing symbols, the Arabic system is still superior, because after sufficient learning Arabic symbols can be easily read and written.

At the level of bases, the base 10 of the Arabic system is a manageable size. There is a trade-off in base size: larger bases require more symbols to be learned but are more efficient in handling large numbers. The addition and multiplication tables that need to be memorized for the Arabic system is relatively small in comparison with the Babylonian system (base 60) and the Mayan system (base 20), which also use positions to represent their main power dimensions.

At the level of symbols, the ten symbols of the Arabic system are easy to write, which makes calculation with paper and pencil efficient.

Combining the representational properties at these four levels, the Arabic system is better than many other systems in terms of representation. Though the Arabic system is also better than many other systems (e.g., Greek, Egyptian, etc.) in terms of calculation, it is not always the most efficient one. As discussed above, the abacus, which was still widely used in Japan, China, and Russia before electronic calculators became popular, is more efficient than the Arabic system for addition and subtraction.

So what makes the Arabic system so special? Numerals have two major functions: representation and calculation. In many cultures, these two functions are achieved by two separate systems. For example, in China, calculation was carried out by abacuses and sticks, whereas representation was realized by written numerals. A more interesting case is the Roman counting board, in which counters were used for calculation and Roman numerals were used for representation, both in the same physical device. We propose that what makes the Arabic system so special and widely accepted is that it integrates representation and calculation into a single system, in addition to its other nice features of efficient information encoding, compactness, extendibility, spatial representation, small base, effectiveness of calculation, and especially important, ease of writing. Thus, Arabic numerals are a special type of object-symbols (Hutchins & Norman, 1988; Norman, 1991)—symbols that serve the functions of both representation and manipulation. The integration of representation and calculation into a single system, however, would be useless without appropriate media and tools such as paper and pencil: a nice example of technological constraints. Interestingly, calculation and the Arabic numerals were so integrated that the word *algorithm* is merely a corruption of Al Kworesmi, the name of the Arabic mathematician of the ninth century whose book on Arabic numerals was the first one reaching Western Europe.

Algebra, in the broad sense in which the term is used today, deals with operations upon symbolic forms, including numerals, unknowns, and arithmetic operators. It has been argued (e.g., Dantzig, 1939) that the invention of Arabic numerals was instrumental in the emergence and development of algebra. Our analysis of Arabic numerals supports this argument: Arabic numerals are not only an efficient representation of numbers but also a symbolic form that can be easily operated upon. It is also interesting to note that Greeks, who were highly advanced in geometry, did not develop an algebra in the modern sense. This is probably because Greek alphabetical numerals, though easy to manipulate, were neither efficient for representation nor for calculation.

Implications for Other Representational Systems

Although our present study has focused on numeration systems, the methodology of representational analysis and the principles of distributed representations illustrated here and elsewhere (Zhang & Norman, 1994a; Zhang, 1992, 1995) can be applied to other domains as well. For example, relational information displays, which include graphs, charts, plots, diagrams, tables, lists, and other types of visual displays that represent relational information, are distributed representations; and the tasks for relational information displays are distributed cognitive tasks. Zhang & Norman (1994b) analyzed relation information displays in terms of a hierarchical structure similar to that of numeration systems and developed a cognitive taxonomy that can classify nearly all types of relational information displays and can serve as a theoretical framework for systematic studies of the cognitive properties in relational information displays. As another example, several cockpit instrument displays, such as different types of altimeters and navigation displays, were also studied from the same perspective (Zhang, 1992).

In addition to numeration systems, numerous representational systems (usually called notational systems) have been invented over the last few thousand years in the development of mathematics (see the two volume book by Cajori, 1928, on the history of mathematical notations). Of these systems, only a fraction survived. Although political, economic, social, and cultural factors certainly played some roles in the evolution of representational systems, cognitive factors might have played the most important role since the activities of individuals in mathematics are essentially cognitive activities. How the forms of representations affect the cognitive activities of scientists and what roles they played in the evolution of mathematics in particular and science in general are interesting issues worth of the attention of not just historians and philosophers but also psychologists and cognitive scientists.

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